Phys 410
Fall 2014
Lecture #3 Summary
9 September, 2014

We discussed the motion of a charged particle in a uniform and uni-directional magnetic field $\vec{B}$, subject to the Lorentz force$\vec{F} = q\vec{v} \times \vec{B}$, where q is the charge of the particle. We took $\vec{B} = B\hat{z}$ and found that Newton's second law of motion reduces to three scalar equations: $m\dot{v}_x = qv_yB$, $m\dot{v}_y = -qv_xB$, and $m\dot{v}_z = 0$. The solution for the motion along the magnetic field direction is simple: $z(t) = z_0 + v_{z0}t$, which is uniform motion at constant velocity. We solved the x-y plane motion using the trick of mapping this two-dimensional problem into the complex plane. Define the complex variable $\eta \equiv v_x + iv_y$, where $i = \sqrt{-1}$. The velocity of the particle is now represented as a point in the complex η plane, and the solution for the velocity evolution with time is a trajectory in the complex η plane. The pair of coupled differential equations now reduces to a simple equation for the time evolution of η, namely $\dot{\eta} = -i\omega\eta$, and the Cyclotron frequency is defined as $\omega = qB/m$, for the charged particle of mass m.

The equation is solved as $\eta = \eta_0 e^{-i\omega t}$, where $\eta_0 = v_{x0} + iv_{y0} \equiv v_0 e^{i\delta}$. This equation represents uniform circular motion in the η-plane on a circle of radius v_0 starting at an angle δ and rotating clockwise with angular velocity ω. The initial velocities are related to v_0 and δ as $v_{x0} = v_0 cos\delta$ and $v_{y0} = v_0 sin\delta$, and $v_0 = \sqrt{v_{x0}^2 + v_{y0}^2}$, $\delta = \tan^{-1}\left(v_{y0}/v_{x0}\right)$. The resulting description of the motion can be obtained by taking the real and imaginary parts of η as $v_x(t) = Re[\eta] = v_0\cos(\delta - \omega t)$, and $v_y(t) = Im[\eta] = v_0\sin(\delta - \omega t)$.

The trajectory of the particle in the xy-plane can be solved by a similar method. First define the complex variable $\xi \equiv x + iy$, and relate it to η through the time derivative: $\eta = \dot{\xi}$. Integrate this equation and apply the initial conditions for x and y to obtain $\xi(t) = r_0 e^{i(\phi_0 - \omega t)}$, where the initial positions are written as $x_0 + iy_0 = r_0 e^{i\phi_0}$. The particle motion is described by uniform circular motion around a circle of radius r_0 starting at angle ϕ_0 at angular velocity ω. The resulting motion is three dimensions is helical about the magnetic field (z) axis.

We considered several applications of these ideas to the cyclotron, the Calutron, and Whistlers in the magneto-sphere of the earth.

We recalled the definition of the total momentum $\vec{P}$ of a many particle system as simply the sum over all the particles of the elementary momentum of each particle, $\vec{P} = \sum_{\alpha=1}^{N} \vec{p}_\alpha = \sum_{\alpha=1}^{N} m_\alpha \vec{v}_\alpha$. If the particles in the system interact with each other by means of forces that obey Newton's third law of motion, the change in total momentum is simply the result of a net external force: $\dot{\vec{P}} = \vec{F}_{net}^{ext}$. This is a generalization of Newton's second law of motion to extended systems. An important consequence is that if the net external force is zero, then the total momentum of the many-particle system is conserved. This is true independent of the nature of the forces between the particles in the system, be they electromagnetic, nuclear, conservative or non-conservative (i.e. forces that convert mechanical energy in to 'heat').

As an example of momentum conservation of a many-particle system with non-conservative forces between the particles, we considered a rocket in free space, subject to zero net external force. It can begin to move by ejecting mass at a speed v_{ex} relative to the rocket. By conservation of momentum, the rocket gains an equal and opposite moment to that given to the ejected fuel. While describing the momentum of the rocket + exhaust from an inertial reference frame we found that $m\dot{v} = -v_{ex}\dot{m}$, where v is the speed of the rocket, m is its mass, and $\dot{m}$ is the rate at which it is ejecting mass. The thrust force on the rocket is $-v_{ex}\dot{m}$. We also found an expression for the net change in velocity of the rocket as $v - v_0 = v_{ex} \ln \frac{m_0}{m}$, where m_0 is the initial mass and m is the final mass. In order to maximize the rocket velocity one should maximize the exhaust speed v_{ex} and the ratio $\frac{m_0}{m}$. The exhaust speed typically depends on the violent exothermic chemical reaction that takes place in the rocket motor, making space flight fairly dangerous.